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Cumulative Sum Control Charts for Marshall-Olkin Extended Exponential Distribution

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ABSTRACT
Cumulative Sum Control Charts (CSCC) is used to maintain current control by continuous inspection of a manufacturing process. Khaparde and Dhabe (2010) developed the Control Charts For Random Queue Length N for $(M/M/1) : (\infty/FCFS)$ Queuing Model. Therefore, in this paper, CSCC has been developed for Marshall-Olkin Extended Exponential (MOEE) Distribution considering quality High way Road Construction with application of Queuing Theory. These charts are useful where quality characteristics are measured by product life time. The expression for Average Run Length (ARL) has been derived. Numerical examples have been included to illustrate the mathematical findings. Some martingales related to cumulative sum tests and single-server queue has been discussed.

KEYWORDS
CSCC; Marshall-Olkin Extended Distribution; Queuing Theory; ARL.

1. Introduction

The concept of Cumulative Sum Control Charts (CSCC) to maintain current control by continuous inspection of a manufacturing process introduced by Page (1954). Barnard (1959) proposed CSCC techniques by treating the manufacturing process as a stochastic process. For this, he used the theory of runs and suggested that a V-mask superimposed on the cu-sum chart be used. Later on, Johnson (1961) showed that CSCC could be interpreted as a modified form of a pair of Sequential Probability Ratio (SPR) tests, operated simultaneously. The idea behind this was the sequential test for two alternative hypotheses. He also gave the procedure for determining the parameters of the V-mask viz. the lead distance d and angle q between the arms of the V-mask and the horizontal line. He worked out details for checking the mean of a normal process and illustrated the use for folded normal. Johnson (1959) developed the cumulative sum control charts for controlling the scale parameters (mean) of a two-parameter Weibull distribution, assuming the shape parameter to be known. He further, showed that the cumulative sum control charts for controlling the mean of an exponentially distributed variable can also be obtained as a special case. Gosh

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(1982) established CSCC for the normal distribution under Type-I censoring. Singh and Shankar (1983) studied CSCC for the mean of an exponential population using extremes. Some references may be made due to Shuguang et al. (2012), Muazu et al. (2013), Nasir et al. (2013), Mahmud and Maravelakis (2011), Philippe and Petros (2011), Abdul et al. (2014), Sheng and Zhang (2026), Saghaei et al. (2009), Alaa and Essam (2009), Jiao and Helo(2006), , Shankar and Gangeshwer (2005, 2006), Zhang and Chen (2015), and Kumar (2016) etc.

Kakoty and Chakraborty (1987) proposed cumulative sum control scheme for truncated exponential distribution with grouped observations. Later on, Shankar and Joseph (1996) studied CSCC for Inverse Gaussian distribution under type-I censoring. Shankar and Gangeshwer (2005) developed the cumulative sum control chart retentive production processes. Chen and Shiyu (2015) proposed the cumulative sum (CUSUM) schemes to efficiently monitor the performance of typical queueing systems based on different sampling schemes. We use M/M/1 queues to illustrate how to design the CUSUM chart and compare their performance with several alternative methods. Olteanu (2010) studied the likelihood ratio based cumulative sum (CUSUM) control charts for censored lifetime data with non-normal distributions.

Shore (2006) studied the Control charts for the queue length in a G/G/S system. Khaparde and Dhabe (2010) developed the Control Charts For Random Queue Length N for $(M/M/1) : (\infty/FCFS)$ Queuing Model. Later on, Zhao and Gilbert (2015) prepared a statistical control chart for monitoring customer waiting time. Queuing theory is an important field of applied mathematics and its origin can be traced to the starting of twentieth century. It was originated to provide and construct the models to predict the behavior of systems. There are various applications of the queuing theory in the literature of probability, operations research, engineering, and management science. The main work in the field of queuing theory has been developed by Erlang (1909), an engineer at the Copenhagen telephone exchange, who published the paper 'The theory of probabilities and telephone conversations'. Later on Fry (1928) in his book on 'Probability and its engineering uses' threw further light on queue-related problems. Landmark work on networks of queues was done by Jackson (1957). Gaver (1959) made a successful attempt to solve queuing problem using imbedded Markov chain technique assuming time space to be continuous.

Further, Rao et al. (2009) studied reliability test plans for Marshall-Olkin extended exponential distribution Lifetime of a product follows Marshall-Olkin extended exponential distribution and it has not been used for the development of CSCC. In this paper, Cumulative Sum Control Charts (CSCC) has been developed for Marshall-Olkin Extended Exponential Distribution considering quality High way Road Construction with Queuing Theory .Johnson's (1961) technique has been utilized to determine the parameters of the mask. The expression for ARL has been derived and illustrated numerically.

2. CSCC for Marshall-Olkin Extended Exponential Distribution

Let the underlying distribution of product life x be extended exponential with parameter inf and σ and the probability density function (P.d.f.) is given by:

$$g(x;\alpha,\sigma) = \frac{\frac{\alpha}{\sigma}\exp^{-x/\sigma}}{(1-\bar{\alpha})^2}, x > 0, \alpha, \sigma > 0, \bar{\alpha} = 1 - \alpha ...(2.1)$$

Let n items be put on test for a pre-assigned time t, called the truncation time, and let the number of failures occurring on or before time t be r. Let us further define a random variable y_i, such that; If the ith item survives beyond time t, then,

$$y_i = 0$$

. if an item fails on or before time t, then,

$$y_i = 1...(2.2)$$

. The total number of failures during time t is given by,

$$r = \sum_{i=1}^{n} y_i...(2.3)$$

The probability of any item failing on or before t is given by,

$$p = \int_0^t g(x, \alpha, \sigma)dx = \phi(1 - \bar{\alpha}e^{-x/\sigma})...(2.4)$$

where,

$$\phi(x) = \int_0^t \frac{\alpha}{\bar{\alpha}}[log(-\bar{\alpha}) - \frac{t}{\sigma}]...(2.5)$$

Since t and σ are known, in constructing the CSCC, we shall treat p as a function of α evidently, p is monotonically decreasing α in and hence Gupta (1962),

$$p \leq p_0 \iff \alpha \geq \alpha_0...(2.6)$$

p_0 is given by (2.4), for any value $\alpha_0 \leq \alpha$. The idea behind the construction of CSCC using the number of failures occurring on or before a pre-fixed time t, is based on the fact that any hypothesis regarding α can uniquely be expressed in terms of p. Suppose we want to detect an increase in the mean α or equivalently, decrease in the parameter p.

Then, following Johnson (1961), we have to consider the SPR tests for discriminating between the following hypotheses,

$$H_0 : \alpha = \alpha_0$$

and

$$H_1 : \alpha = \alpha_1(\alpha_1 > \alpha_0).$$

or,

$$H_0 : p = p_0$$

and

$$H_1 : p = p_1, (p_1 < p_0)...(2.7)$$

Clearly, the SPR test discriminating between the above two hypotheses has as its continuation region,

$$\frac{\log\{\beta_1/(1-\beta_0)\} + n\log\left\{\dfrac{1-p_1}{1-p_0}\right\}}{\log\left\{\dfrac{p_0(1-p_1)}{p_1(1-p_0)}\right\}} < r < \frac{\log\{(1-\beta_1)/\beta_0\} + n\log\left\{\dfrac{1-p_1}{1-p_0}\right\}}{\log\left\{\dfrac{p_0(1-p_1)}{p_1(1-p_0)}\right\}}. \tag{2.8}$$

Where β_0 and β_1 are the probabilities of type I and type II errors respectively.
Considering only right hand side inequality in (2.8) with $\beta_1 = 0$, we get,

$$r > \frac{-\log\beta_0 + n\log\left\{\dfrac{1-p_1}{1-p_0}\right\}}{\log\left\{\dfrac{p_0(1-p_1)}{p_1(1-p_0)}\right\}}. \tag{2.9}$$

as the inequality for constructing the CSCC for detecting an increase in process mean. The control limit PQ in Figure (1.a) is obtained by plotting the points with co-ordinates (n, r) so that any plotted points above the line PQ is an evidence of lack of control i.e. an indication of increase in p equivalently a decrease in ∞. The lead distance OP, where O is the last plotted point and P is the vertex, and slope of the line PQ with the horizontal axis as,

$$\text{OP} = \frac{-\log\beta_0}{\log\left\{\dfrac{1-p_1}{1-p_0}\right\}}. \tag{2.10}$$

$$\tan\angle O'PQ = \frac{\log\left\{\dfrac{1-p_1}{1-p_0}\right\}}{\log\left\{\dfrac{p_0(1-p_1)}{p_1(1-p_0)}\right\}}. \tag{2.11}$$

Now, for detecting decrease in α from α_0 to α_2, we have to consider the following hypotheses,

$$H_0 : p = p_0 \quad \text{and} \quad H_2 : p = p_2, \qquad (p_2 > p_0). \tag{2.12}$$

and proceeding in a similar manner as above, we obtain the inequality

$$r < \frac{-\log\beta_0 + n\log\left\{\dfrac{1-p_0}{1-p}\right\}}{\log\left\{\dfrac{p_2(1-p_0)}{p_0(1-p_2)}\right\}}. \tag{2.13}$$

The control limit $P'Q'$ as shown in Figure (1.b) can be obtained by plotting the points with co-ordinates (n, r). The parameters of the control chart are given by

$$\mathrm{OP}' = \frac{-\log\beta_0}{\log\left\{\dfrac{1-p_0}{1-p_2}\right\}}. \tag{2.14}$$

$$\tan\angle OP'Q' = \frac{\log\left\{\dfrac{1-p_0}{1-p_2}\right\}}{\log\left\{\dfrac{p_2(1-p_0)}{p_0(1-p_2)}\right\}}. \tag{2.15}$$

Any plotted points lying below $P'Q'$ will indicate a lack of control.

A two-sided CSCC can now easily be obtained by considering simultaneously a pair of SPR tests, as proposed by Armitage (1950) with the first kind of error $2\beta_0$ and the second kind of error $\beta_1 \cong 0$, as usual. The control-chart diagrams are then shown in Fig. (2) with parameters given by (2.10), (2.11), (2.14) and (2.15). Any points outside the arms of the chart will indicate a lack of control.

3. Average Run Length (ARL)

The approximate formula for ARL, when H_i $(i = 1, 2)$ is true, is given by Johnson (1961) as

$$\mathrm{ARL} = (-\log\beta_0)E^{-1}. \tag{3.1}$$

where

$$E = \mathrm{E}\left[\log\left\{\frac{f(y/p_i)}{f(y/p_0)}\right\} \Big/ H_i\right]. \tag{3.2}$$

Using (3.1) and (3.2) the approximate formula of the ARL can be written as

$$\mathrm{ARL} \cong \frac{-\log\beta_0}{p_i\log\left(\dfrac{p_i}{p_0}\right) + (1-p_i)\log\left\{\dfrac{1-p_i}{1-p_0}\right\}}. \tag{3.3}$$

The approximate formula for the ARL of a two-sided CSCC can be obtained as

$$(\text{ARL}) = (\text{ARL}/H_1)^{-1} + (\text{ARL}/H_2)^{-1}. \tag{3.4}$$

Where (ARL/H_i) implies the ARL, when H_i $(i = 1, 2)$ is true.

4. Illustration and Discussion of Results

Suppose failures are noted for item being tested for time $t = 50$ months. It is desired to maintain the mean-life at first stage $\alpha_0 = 250$ months. For simplicity, it is assumed that only decrease in the mean-life is undesirable, so that a one-sided cu-sum chart $\alpha_1 > \alpha_2$ is to be used. It is desired to test

$$H_0: \ \alpha_1 = \alpha_0 = 250 \text{ months}, \qquad H_2: \ \alpha_1 = \alpha_2 = 100 \text{ months}.$$

It is further assumed that the distribution of life is Marshall-Olkin Extended Exponential (MOEE) distribution with known standard deviation. Setting $\beta_0 = 0.05$, $\alpha = 2$ and choosing $t/\alpha_0 = 50/250 = 0.20$, $t/\alpha_2 = 50/100 = 0.50$, we find $p_0 = 0.197$ and $p_1 = 0.078$. The lead distance is OP = 21, by using Equation (2.10). The slope $\tan\angle OP'Q' = 0.13$, so that $\angle OP'Q' = 7^\circ 40'$ by using Equation (2.11). The ARL value comes out to be $54.43 \approx 54$. These results are on the line of Shankar and Joseph (1996), $31.87 \approx 32$, and Shankar and Gangeshwer (2005), $24.02 \approx 24$. It seems inconvenient to work with such a small angle, so the vertical scale of measurement can be reduced to obtain a workable slope. For example, if the unit along the vertical scale is reduced to one-tenth of the original unit, then we have

$$\tan\angle OP'Q' = 1.3,$$

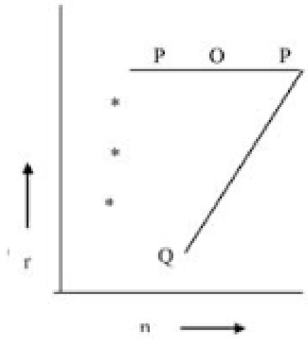
so that

$$\angle OP'Q' = 52^\circ 43'.$$

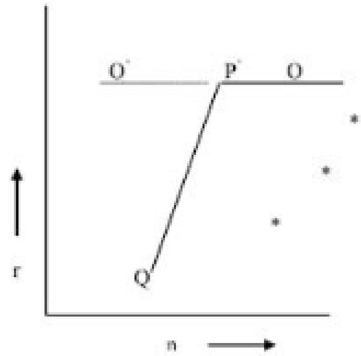
In this paper, CSCC has been developed for the Marshall–Olkin Extended Exponential distribution under Type-I censoring, and the results agree with those reported by Ghosh (1982).

Montgomery (1992) remarked that quality and productivity improvement has become an essential element of the overall strategic plan for most organizations. This has sparked renewed interest in statistical methods for quality improvement. Dickinson et al. (2014) explained that the lifetime of the product is a key characteristic of product quality and a common concern to both engineers and statisticians. It designates the time span during which a product can be expected to operate safely and meet specified performance standards. In this paper, CSCC has been developed for the Marshall–Olkin Extended Exponential distribution under the assumption of Type-I censoring. This type of situation may frequently be encountered in life tests, fatigue tests, or other destructive tests, where it is customary as well as advantageous to truncate the experiment at a pre-assigned time, say t, and draw inferences on the basis of the number of failures occurring on or before time t. These charts are useful where quality characteristics are measured by product lifetime.

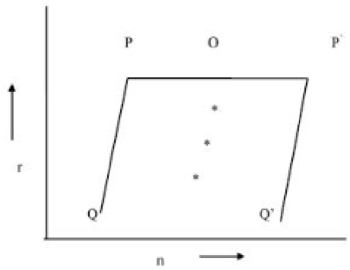
Johnson's (1961) technique has been utilized to determine the parameters of the mask. The expression for ARL has been derived and illustrated numerically. However, the comparison can be made with EWMA, adaptive CUSUM, GLR-CUSUM, and Bayesian CUSUM. Moreover, Kennedy (1976) has applied CUSUM charts in queueing processes to compute the distribution of the first passage times for an M/M/1 queue. CUSUM charts have also been used to calculate stopping times associated with sequential cumulative sum tests in health care and public health. Lawrance and Lewis (1977) studied exponential moving average processes of the first order. These processes are important in queueing and network problems while constructing highways. The queueing models developed may also contribute enormously to the validation and evolution of economic problems. The present solution would be useful for researchers and quality control professionals.



Fig(1.a), CSCC for detecting the decrease in the process mean ($p_2 > p_0$)



Fig(1.b), CSCC for detecting the increase in the process mean ($p_1 < p_0$)



Fig(2) Two sided CSCC for the mean of Marshall-Olkin extended exponential distribution

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